ON SEPARATION OF VARIABLES AND COMPLETENESS OF THE BETHE ANSATZ FOR QUANTUM $\mathfrak{gl}_N$ GAUDIN MODEL

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Abstract. In this note, we discuss implications of the results obtained in [MTV4]. It was shown there that eigenvectors of the Bethe algebra of the quantum $\mathfrak{gl}_N$ Gaudin model are in a one-to-one correspondence with Fuchsian differential operators with polynomial kernel. Here, we interpret this fact as a separation of variables in the $\mathfrak{gl}_N$ Gaudin model. Having a Fuchsian differential operator with polynomial kernel, we construct the corresponding eigenvector of the Bethe algebra. It was shown in [MTV4] that the Bethe algebra has simple spectrum if the evaluation parameters of the Gaudin model are generic. In that case, our Bethe ansatz construction produces an eigenbasis of the Bethe algebra.

1. Introduction

Generally speaking, separation of variables in a quantum integrable model is a reduction of a multidimensional spectral problem to a suitable one-dimensional problem. For example, the famous Sklyanin's separation of variables for the $\mathfrak{gl}_2$ Gaudin model [Sk] is a reduction of the diagonalization problem of the Gaudin Hamiltonians, acting on a tensor product of $\mathfrak{gl}_2$-modules, to the problem of finding a second order Fuchsian differential operator with polynomial kernel and prescribed singularities. Having such a differential operator, Sklyanin constructs an eigenvector of the Hamiltonians.

It has been proved recently in [MTV4] that the eigenvectors of the Bethe algebra of the $\mathfrak{gl}_N$ Gaudin model are in a bijective correspondence with N-th order Fuchsian differential operators with polynomial kernel and prescribed singularities. This reduces the multidimensional problem of the diagonalization of the Bethe algebra to the one-dimensional problem of finding the corresponding Fuchsian differential operators. In that respect, "the variables are separated".

Having an eigenvector of the Bethe algebra, one has an effective way to construct the corresponding Fuchsian operator, see [MTV2], [MTV4], and Theorem 2.1. In the opposite direction, the assignment of an eigenvector to a Fuchsian operator is not explicit in [MTV4]. In this note, having a Fuchsian differential operator with polynomial kernel, we construct the corresponding eigenvector of the Bethe algebra. Our construction of an eigenvector from

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a differential operator can be viewed as a (generalized) Bethe ansatz construction, cf. [Ba], [RV], [MV1], [MV2].

It has been proved in [MTV4], that the action of the Bethe algebra on a tensor product of irreducible finite-dimensional evaluation $\mathfrak{gl}_N[t]$-modules has simple spectrum provided the evaluation points are generic. In that case, our construction of eigenvectors of the Bethe algebra produces an eigenbasis of the Bethe algebra, thus showing the completeness of the Bethe ansatz.

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2. Eigenvectors of Bethe algebra

2.1. Lie algebra $\mathfrak{gl}_N$. Let e_{ij}, $i, j = 1, \dots, N$, be the standard generators of the Lie algebra $\mathfrak{gl}_N$ satisfying the relations $[e_{ij}, e_{sk}] = \delta_{js} e_{ik} - \delta_{ik} e_{sj}$.

Let M be a $\mathfrak{gl}_N$-module. A vector $v \in M$ has weight $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$ if $e_{ii} v = \lambda_i v$ for $i = 1, \dots, N$. A vector v is called *singular* if $e_{ij} v = 0$ for $1 \leqslant i < j \leqslant N$.

We denote by $(M)_{\boldsymbol{\lambda}}$ the subspace of M of weight $\boldsymbol{\lambda}$, by $(M)^{\mathrm{sing}}$ the subspace of M of all singular vectors and by $(M)^{\mathrm{sing}}_{\boldsymbol{\lambda}}$ the subspace of M of all singular vectors of weight $\boldsymbol{\lambda}$.

Denote by $L_{\boldsymbol{\lambda}}$ the irreducible finite-dimensional $\mathfrak{gl}_N$-module with highest weight $\boldsymbol{\lambda}$. Any finite-dimensional $\mathfrak{gl}_N$-module M is isomorphic to the direct sum $\bigoplus_{\boldsymbol{\lambda}} L_{\boldsymbol{\lambda}} \otimes (M)^{\mathrm{sing}}_{\boldsymbol{\lambda}}$, where the spaces $(M)^{\mathrm{sing}}_{\boldsymbol{\lambda}}$ are considered as trivial $\mathfrak{gl}_N$-modules.

The $\mathfrak{gl}_N$-module $L_{(1,0,\dots,0)}$ is the standard N-dimensional vector representation of $\mathfrak{gl}_N$. We denote it by V. We choose a highest weight vector in V and denote it by v_+.

A $\mathfrak{gl}_N$-module M is called polynomial if it is isomorphic to a submodule of $V^{\otimes n}$ for some n.

A sequence of integers $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_N \geqslant 0$ is called a *partition with at most N parts*. Set $|\boldsymbol{\lambda}| = \sum_{i=1}^N \lambda_i$. Then it is said that $\boldsymbol{\lambda}$ is a partition of $|\boldsymbol{\lambda}|$.

The $\mathfrak{gl}_N$-module $V^{\otimes n}$ contains the module $L_{\boldsymbol{\lambda}}$ if and only if $\boldsymbol{\lambda}$ is a partition of n with at most N parts.

2.2. Current algebra $\mathfrak{gl}_N[t]$. Let $\mathfrak{gl}_N[t] = \mathfrak{gl}_N \otimes \mathbb{C}[t]$ be the Lie algebra of $\mathfrak{gl}_N$-valued polynomials with the pointwise commutator. We call it the *current algebra*. We identify the Lie algebra $\mathfrak{gl}_N$ with the subalgebra $\mathfrak{gl}_N \otimes 1$ of constant polynomials in $\mathfrak{gl}_N[t]$. Hence, any $\mathfrak{gl}_N[t]$-module has the canonical structure of a $\mathfrak{gl}_N$-module.

It is convenient to collect elements of $\mathfrak{gl}_N[t]$ in generating series of a formal variable u. For $g \in \mathfrak{gl}_N$, set

$$g(u) = \sum_{s=0}^{\infty} (g \otimes t^s) u^{-s-1}.$$

For each $a \in \mathbb{C}$, there exists an automorphism ρ_a of $\mathfrak{gl}_N[t]$, $\rho_a : g(u) \mapsto g(u-a)$. Given a $\mathfrak{gl}_N[t]$-module M, we denote by $M(a)$ the pull-back of M through the automorphism ρ_a. As $\mathfrak{gl}_N$-modules, M and $M(a)$ are isomorphic by the identity map.

We have the evaluation homomorphism, $ev : \mathfrak{gl}_N[t] \to \mathfrak{gl}_N$, $ev : g(u) \mapsto g\, u^{-1}$. Its restriction to the subalgebra $\mathfrak{gl}_N \subset \mathfrak{gl}_N[t]$ is the identity map. For any $\mathfrak{gl}_N$-module M, we denote

by the same letter the $\mathfrak{gl}_N[t]$-module, obtained by pulling M back through the evaluation homomorphism. For each $a \in \mathbb{C}$, the $\mathfrak{gl}_N[t]$-module $M(a)$ is called an *evaluation module*.

2.3. **Bethe algebra.** Given an $N \times N$ matrix A with possibly noncommuting entries a_{ij}, we define its *row determinant* to be

$$\operatorname{rdet} A \,=\, \sum_{\sigma \in S_N} (-1)^\sigma \, a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{N\sigma(N)} \,.$$

Let ∂ be the operator of differentiation in variable u. Define the *universal differential operator* $\mathcal{D}^{\mathcal{B}}$ by

$$\mathcal{D}^{\mathcal{B}} \,=\, \operatorname{rdet} \begin{pmatrix} \partial - e_{11}(u) & -\,e_{21}(u) & \ldots & -\,e_{N1}(u) \\ -\,e_{12}(u) & \partial - e_{22}(u) & \ldots & -\,e_{N2}(u) \\ \ldots & \ldots & \ldots & \ldots \\ -\,e_{1N}(u) & -\,e_{2N}(u) & \ldots & \partial - e_{NN}(u) \end{pmatrix}.$$

It is a differential operator in variable u, whose coefficients are formal power series in u^{-1} with coefficients in $U(\mathfrak{gl}_N[t])$,

$$\mathcal{D}^{\mathcal{B}} \,=\, \partial^N + \sum_{i=1}^{N} B_i(u)\, \partial^{N-i},$$

where

$$B_i(u) \,=\, \sum_{j=i}^{\infty} B_{ij}\, u^{-j}\,,$$

and $B_{ij} \in U(\mathfrak{gl}_N[t])$, $i = 1, \ldots, N$, $j \in \mathbb{Z}_{\geqslant i}$. We call the unital subalgebra of $U(\mathfrak{gl}_N[t])$ generated by B_{ij}, $i = 1, \ldots, N$, $j \in \mathbb{Z}_{\geqslant i}$, the *Bethe algebra* and denote it by $\mathcal{B}$.

By [T], [MTV1], the algebra $\mathcal{B}$ is commutative, and the algebra $\mathcal{B}$ commutes with the subalgebra $U(\mathfrak{gl}_N) \subset U(\mathfrak{gl}_N[t])$.

As a subalgebra of $U(\mathfrak{gl}_N[t])$, the algebra $\mathcal{B}$ acts on any $\mathfrak{gl}_N[t]$-module M. Since $\mathcal{B}$ commutes with $U(\mathfrak{gl}_N)$, it preserves the subspace of singular vectors $(M)^{\mathrm{sing}}$ as well as weight subspaces of M. Therefore, the subspace $(M)^{\mathrm{sing}}_{\boldsymbol{\lambda}}$ is $\mathcal{B}$-invariant for any weight $\boldsymbol{\lambda}$.

Let $\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(k)}, \boldsymbol{\lambda}$ be partitions with at most N parts, and $b_1, \ldots, b_k$ distinct complex numbers. We are interested in the action of the Bethe algebra $\mathcal{B}$ on the tensor product of evaluation modules $\otimes_{s=1}^k L_{\boldsymbol{\lambda}^{(s)}}(b_s)$, more precisely, on the subspace $(\otimes_{s=1}^k L_{\boldsymbol{\lambda}^{(s)}}(b_s))^{\mathrm{sing}}_{\boldsymbol{\lambda}}$.

Note that the subspace $(\otimes_{s=1}^k L_{\boldsymbol{\lambda}^{(s)}}(b_s))^{\mathrm{sing}}_{\boldsymbol{\lambda}}$ is zero-dimensional unless $|\boldsymbol{\lambda}| = \sum_{s=1}^k |\boldsymbol{\lambda}^{(s)}|$.

2.4. **Fuchsian differential operators and eigenvectors of Bethe algebra.** Denote $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(k)})$ and $\boldsymbol{b} = (b_1, \ldots, b_k)$. Let $\Delta_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ be the set of all monic Fuchsian differential operators of order N,

$$\mathcal{D} \,=\, \partial^N + \sum_{i=1}^{N} h_i^{\mathcal{D}}(u)\, \partial^{N-i},$$

with the following properties.

a) The singular points of $\mathcal{D}$ are at $b_1, \ldots, b_k$ and ∞ only.

b) The exponents of $\mathcal{D}$ at b_s, $s = 1, \ldots, k$, are equal to $\lambda^{(s)}_N$, $\lambda^{(s)}_{N-1} + 1, \ldots, \lambda^{(s)}_1 + N - 1$.

c) The exponents of $\mathcal{D}$ at ∞ are equal to $1-N-\lambda_1,\, 2-N-\lambda_2,\, \dots\, , -\lambda_N$.
d) The kernel of the operator $\mathcal{D}$ consists of polynomials only.

Note that the set $\Delta_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ is empty unless $|\boldsymbol{\lambda}| = \sum_{s=1}^{k} |\boldsymbol{\lambda}^{(s)}|$.

Let M be a $\mathfrak{gl}_N[t]$-module and v an eigenvector of the Bethe algebra $\mathcal{B} \subset U(\mathfrak{gl}_N[t])$ acting on M. Then for any coefficient $B_i(u)$ of the universal differential operator $\mathcal{D}^{\mathcal{B}}$ we have $B_i(u)v = h_i(u)v$, where $h_i(u)$ is a scalar series. We call the scalar differential operator

$$\mathcal{D}^{\mathcal{B}}_v \,=\, \partial^N + \sum_{i=1}^{N} h_i(u)\, \partial^{N-i}$$

the *differential operator associated with the eigenvector* v.

Theorem 2.1. *Let* $v \in (\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s))^{\mathrm{sing}}_{\boldsymbol{\lambda}}$ *be an eigenvector of the Bethe algebra, then* $\mathcal{D}^{\mathcal{B}}_v \in \Delta_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$. *Moreover, the assignment* $v \mapsto \mathcal{D}^{\mathcal{B}}_v$ *is a bijective correspondence between the set of eigenvectors of the action of the Bethe algebra on* $(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s))^{\mathrm{sing}}_{\boldsymbol{\lambda}}$ *(considered up to multiplication by nonzero numbers) and the set* $\Delta_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$.

The first statement is Theorem 4.1 in [MTV2], cf. [MTV3]. The second statement is Theorem 7.1 in [MTV4].

The goal of this note is to construct the inverse bijection.

3. Schubert cell and universal weight function

3.1. **The cell $\Omega_{\boldsymbol{\lambda}}$.** Let $N, d \in \mathbb{Z}_{>0}$, $N \leqslant d$. Let $\mathbb{C}_d[u]$ be the space of polynomials in u of degree less than d. We have $\dim \mathbb{C}_d[u] = d$. Let $\mathrm{Gr}(N,d)$ be the Grassmannian of all N-dimensional subspaces in $\mathbb{C}_d[u]$.

Given a partition $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$ such that $\lambda_1 \leqslant d-N$, introduce a sequence

$$P \,=\, \{d_1 > d_2 > \dots > d_N\}\,, \qquad d_i = \lambda_i + N - i\,,$$

and denote by $\Omega_{\boldsymbol{\lambda}}$ the subset of $\mathrm{Gr}(N,d)$ consisting of all N-dimensional subspaces $X \subset \mathbb{C}_d[u]$ such that for every $i = 1, \dots, N$, the subspace X contains a polynomial of degree d_i.

In other words, $\Omega_{\boldsymbol{\lambda}}$ consists of subspaces $X \subset \mathbb{C}_d[u]$ with a basis $\{f_1(u), \dots, f_N(u)\}$ of the form

$$f_i(u) = u^{d_i} + \sum_{j=1,\ d_i - j \notin P}^{d_i} f_{ij} u^{d_i - j}.$$

For a given $X \in \Omega_{\boldsymbol{\lambda}}$, such a basis is unique. The basis $\{f_1(u), \dots, f_N(u)\}$ will be called the flag basis of the subspace X.

The set $\Omega_{\boldsymbol{\lambda}}$ is a (Schubert) cell isomorphic to an affine space of dimension $|\boldsymbol{\lambda}|$ with coordinate functions f_{ij}.

For $X \in \Omega_{\boldsymbol{\lambda}}$, we denote by $\mathcal{D}_X$ the monic scalar differential operator of order N with kernel X. We call $\mathcal{D}_X$ the *differential operator associated with* X.

3.2. **Generic points of $\Omega_{\boldsymbol\lambda}$.** For $g_1, \dots, g_l \in \mathbb{C}[u]$, introduce the Wronskian by the formula

$$\mathrm{Wr}(g_1(u), \dots, g_l(u)) \,=\, \det \begin{pmatrix} g_1(u) & g_1'(u) & \dots & g_1^{(l-1)}(u) \\ g_2(u) & g_2'(u) & \dots & g_2^{(l-1)}(u) \\ \dots & \dots & \dots & \dots \\ g_l(u) & g_l'(u) & \dots & g_l^{(l-1)}(u) \end{pmatrix}.$$

For $X \in \Omega_{\boldsymbol\lambda}$, let $\{f_1(u), \dots, f_N(u)\}$ be the flag basis of X. Introduce the polynomials $\{y_0(u)\,,\, y_1(u)\,,\, \dots\,,\, y_{N-1}(u)\}$, by the formula

$$y_a(u) \prod_{a<i<j\leqslant N} (\lambda_i - \lambda_j) \,=\, \mathrm{Wr}(f_{a+1}(u), \dots, f_N(u))\,, \qquad a = 0, \dots, N\,.$$

Set

$$l_a \,=\, \sum_{b=a+1}^{N} \lambda_b\,, \qquad a = 0, \dots, N\,. \tag{3.1}$$

Clearly, $l_0 = |\boldsymbol\lambda|$ and $l_N = 0$.

For each $a = 0, \dots, N-1$, the polynomial $y_a(u)$ is a monic polynomial of degree l_a. Denote $t_1^{(a)}, \dots, t_{l_a}^{(a)}$ the roots of the polynomial $y_a(u)$, and

$$\boldsymbol{t}_X \,=\, (t_1^{(0)}, \dots, t_{l_0}^{(0)}, \,\dots\,, t_1^{(N-1)}, \dots, t_{l_{N-1}}^{(N-1)})\,. \tag{3.2}$$

We say that $\boldsymbol{t}_X$ are the *root coordinates* of X.

We say that $X \in \Omega_{\boldsymbol\lambda}$ is *generic* if all roots of the polynomials $y_0(u)\,,\, y_1(u)\,,\, \dots\,,\, y_{N-1}(u)$ are simple and for each $a = 1, \dots, N-1$, the polynomials $y_{a-1}(u)$ and $y_a(u)$ do not have common roots.

If X is generic, then the root coordinates $\boldsymbol{t}_X$ satisfy the Bethe ansatz equations [MV1]:

$$\sum_{j'=1}^{l_{a-1}} \frac{1}{t_j^{(a)} - t_{j'}^{(a-1)}} \,-\, \sum_{\substack{j'=1 \\ j'\neq j}}^{l_a} \frac{2}{t_j^{(a)} - t_{j'}^{(a)}} \,+\, \sum_{j'=1}^{l_{a+1}} \frac{1}{t_j^{(a)} - t_{j'}^{(a+1)}} \,=\, 0\,.$$

Here the equations are labeled by $a = 1, \dots, N-1,\; j = 1, \dots, l_a$.

Conversely, if $\boldsymbol{t} \,=\, (t_1^{(0)}, \dots, t_{l_0}^{(0)}, \,\dots\,, t_1^{(N-1)}, \dots, t_{l_{N-1}}^{(N-1)})$ satisfy the Bethe ansatz equations, then there exists a unique $X \in \Omega_{\boldsymbol\lambda}$ such that X is generic and $\boldsymbol{t}$ are its root coordinates, see (3.2). This X is determined by the following construction, see [MV1]. Set

$$\chi^a(u, \boldsymbol{t}) \,=\, \sum_{j=1}^{l_{a-1}} \frac{1}{u - t_j^{(a-1)}} \,-\, \sum_{i=1}^{l_a} \frac{1}{u - t_j^{(a)}}\,, \qquad a = 1, \dots, N\,.$$

Then

$$\mathcal{D}_X \,=\, \big(\partial - \chi^1(u, \boldsymbol{t})\big)\,\dots\,\big(\partial - \chi^N(u, \boldsymbol{t})\big)\,.$$

Lemma 3.1. *Generic points form a Zariski open subset of* $\Omega_{\boldsymbol\lambda}$.

The lemma follows, for example, from part (i) of Theorem 6.1 in [MV2].

3.3. **Universal weight function.** Let $\boldsymbol{\lambda}$ be a partition with at most N parts. Let $l_0, \dots, l_N$ be the numbers defined in (3.1). Denote $n = l_0$, $l = l_1 + \dots + l_{N-1}$ and $\boldsymbol{l} = (l_1, \dots, l_{N-1})$.

Consider the weight subspace $(V^{\otimes n})_{\boldsymbol{\lambda}}$ of the n-th tensor power of the vector representation of $\mathfrak{gl}_N$ and the space $\mathbb{C}^{l+n}$ with coordinates $\boldsymbol{t} = (t^{(0)}_1, \dots, t^{(0)}_{l_0}, \dots, t^{(N-1)}_1, \dots, t^{(N-1)}_{l_{N-1}})$.

In this section we remind the construction of a rational map $\omega : \mathbb{C}^{l+n} \to (V^{\otimes n})_{\boldsymbol{\lambda}}$, called the *universal weight function*, see [SV].

A basis of $V^{\otimes n}$ is formed by the vectors

$$e_J v = e_{j_1,1} v_+ \otimes \dots \otimes e_{j_n,1} v_+ ,$$

where $J = (j_1, \dots, j_n)$ and $1 \leqslant j_s \leqslant N$ for $s = 1, \dots, N$. A basis of $(V^{\otimes n})_{\boldsymbol{\lambda}}$ is formed by the vectors $e_J v$ such that $\#\{s \mid j_s > i\} = l_i$ for every $i = 1, \dots, N-1$. Such a J will be called $\boldsymbol{l}$-admissible.

The universal weight function has the form

$$\omega(\boldsymbol{t}) = \sum_J \omega_J(\boldsymbol{t})\, e_J v$$

where the sum is over the set of all $\boldsymbol{l}$-admissible J, and the function $\omega_J(\boldsymbol{t})$ is defined below.

For an admissible J, define $S(J) = \{s \mid j_s > 1\}$, and for $i = 1, \dots, N-1$, define

$$S_i(J) = \{ s \mid 1 \leqslant s \leqslant n , \quad 1 \leqslant i < j_s \} .$$

Then $|S_i(J)| = l_i$.

Let $B(J)$ be the set of sequences $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{N-1})$ of bijections $\beta_i : S_i(J) \to \{1, \dots, l_i\}$, $i = 1, \dots, N-1$. Then $|B(J)| = \prod_{a=1}^{N-1} l_a!$.

For $s \in S(J)$ and $\boldsymbol{\beta} \in B(J)$, introduce the rational function

$$\omega_{s,\boldsymbol{\beta}}(\boldsymbol{t}) = \frac{1}{t^{(1)}_{\beta_1(s)} - t^{(0)}_s} \prod_{i=2}^{j_1-1} \frac{1}{t^{(i)}_{\beta_i(s)} - t^{(i-1)}_{\beta_{i-1}(s)}}$$

and define

$$\omega_J(\boldsymbol{t}) = \sum_{\boldsymbol{\beta} \in B(J)} \prod_{s \in S(J)} \omega_{s,\boldsymbol{\beta}} .$$

Example. Let $n = 2$ and $\boldsymbol{l} = (1, 1, 0, \dots, 0)$. Then

$$\omega(\boldsymbol{t}) = \frac{1}{(t^{(2)}_1 - t^{(1)}_1)(t^{(1)}_1 - t^{(0)}_1)}\, e_{3,1} v_+ \otimes v_+ + \frac{1}{(t^{(2)}_1 - t^{(1)}_1)(t^{(1)}_1 - t^{(0)}_2)}\, v_+ \otimes e_{3,1} v_+ .$$

Theorem 3.2. *Let $X \in \Omega_{\boldsymbol{\lambda}}$ be a generic point with root coordinates $\boldsymbol{t}_X$. Consider the value $\omega(\boldsymbol{t}_X)$ of the universal weight function $\omega : \mathbb{C}^{l+n} \to (V^{\otimes n})_{\boldsymbol{\lambda}}$ at $\boldsymbol{t}_X$. Consider $V^{\otimes n}$ as the $\mathfrak{gl}_N[t]$-module $\otimes_{s=1}^n V(t^{(0)}_s)$. Then*

(i) *The vector $\omega(\boldsymbol{t}_X)$ belongs to $(V^{\otimes n})^{\mathrm{sing}}_{\boldsymbol{\lambda}}$.*

(ii) *The vector $\omega(\boldsymbol{t}_X)$ is an eigenvector of the Bethe algebra $\mathcal{B}$, acting on $\otimes_{s=1}^n V(t^{(0)}_s)$. Moreover, $\mathcal{D}^{\mathcal{B}}_{\omega(\boldsymbol{t}_X)} = \mathcal{D}_X$, where $\mathcal{D}^{\mathcal{B}}_{\omega(\boldsymbol{t}_X)}$ and $\mathcal{D}_X$ are the differential operators associated with the eigenvector $\omega(\boldsymbol{t}_X)$ and the point $X \in \Omega_{\boldsymbol{\lambda}}$, respectively.*

Part (i) is proved in [Ba] and [RV]. Part (i) also follows directly from Theorem 6.16.2 in [SV]. Part (ii) is proved in [MTV1].

Remark. For a generic point $X \in \Omega_{\boldsymbol{\lambda}}$ the differential operator $\mathcal{D}_X$ has the following properties.

e) The singular points of $\mathcal{D}_X$ are at $t_1^{(0)}, \dots, t_n^{(0)}$ and ∞ only;
f) The exponents of $\mathcal{D}_X$ at $t_s^{(0)}$, $s = 1, \dots, n$, are equal to $0, 1, \dots, N-2, N$;
g) The exponents of $\mathcal{D}_X$ at ∞ are equal to $1-N-\lambda_1, 2-N-\lambda_2, \dots, -\lambda_N$;
h) The kernel of the operator $\mathcal{D}_X$ consists of polynomials only.

On the other hand, Theorem 2.1, applied to the $\mathfrak{gl}_N[t]$-module $\otimes_{s=1}^n V(t_s^{(0)})$, yields that for any eigenvector v of the Bethe algebra $\mathcal{B}$, acting on $(\otimes_{s=1}^n V(t_s^{(0)}))_{\boldsymbol{\lambda}}^{\mathrm{sing}}$, the differential operator $\mathcal{D}_v^{\mathcal{B}}$ has properties e)−h).

Therefore, the universal weight function and the assignment $\mathcal{D}_X \mapsto X \mapsto \omega(\boldsymbol{t}_X)$ allows us to reverse the correspondence $v \mapsto \mathcal{D}_v^{\mathcal{B}}$ of Theorem 2.1 for the case of the $\mathfrak{gl}_N[t]$-module $\otimes_{s=1}^n V(t_s^{(0)})$ under the condition that $X \in \Omega_{\boldsymbol{\lambda}}$ is generic. Our goal is to generalize this construction to the case of a $\mathfrak{gl}_N[t]$-module $\otimes_{s=1}^k L_{\boldsymbol{\lambda}^{(s)}}(b_s)$ and an arbitrary differential operator $\mathcal{D} \in \Delta_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$.

4. Construction of an eigenvector from a differential operator

4.1. **Epimorphism $F_{\boldsymbol{\lambda}}$.** Let $\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(k)}, \boldsymbol{\lambda}$ be partitions with at most N parts such that $|\boldsymbol{\lambda}| = \sum_{s=1}^k |\boldsymbol{\lambda}^{(s)}|$, and $b_1, \dots, b_k$ distinct complex numbers. Denote $n = |\boldsymbol{\lambda}|$ and $n_s = |\boldsymbol{\lambda}^{(s)}|$, $s = 1, \dots, k$.

For $s = 1, \dots, k$, let $F_s : V^{\otimes n_s} \to L_{\boldsymbol{\lambda}^{(s)}}$ be an epimorphism of $\mathfrak{gl}_N$-modules. Then

$$F_1 \otimes \dots \otimes F_k : \otimes_{s=1}^k V(b_s)^{\otimes n_s} \to \otimes_{s=1}^k L_{\boldsymbol{\lambda}^{(s)}}(b_s) \tag{4.1}$$

is an epimorphism of $\mathfrak{gl}_N[t]$-module, which induces an epimorphism of $\mathcal{B}$-modules

$$F : (\otimes_{s=1}^k V(b_s)^{\otimes n_s})_{\boldsymbol{\lambda}}^{\mathrm{sing}} \to (\otimes_{s=1}^k L_{\boldsymbol{\lambda}^{(s)}}(b_s))_{\boldsymbol{\lambda}}^{\mathrm{sing}}.$$

4.2. **Main result.** Let $\mathcal{D}^0$ be an element of $\Delta_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$. Let X^0 be the kernel of $\mathcal{D}^0$. Then X^0 is a point of the cell $\Omega_{\boldsymbol{\lambda}}$. Choose a germ of an algebraic curve $X(\epsilon)$ in $\Omega_{\boldsymbol{\lambda}}$ such that $X(0) = X^0$ and $X(\epsilon)$ are generic points of $\Omega_{\boldsymbol{\lambda}}$ for all nonzero ϵ. Let $\boldsymbol{t}(\epsilon)$ be the root coordinates of $X(\epsilon)$. The algebraic functions $t_1^{(0)}(\epsilon), \dots, t_n^{(0)}(\epsilon)$ are determined up to permutation. Order them in such a way that the first n_1 of them tend to b_1 as $\epsilon \to 0$, the next n_2 coordinates tend to b_2, and so on until the last n_k coordinates tend to b_k.

For every nonzero ϵ, the vector $v(\epsilon) = \omega(\boldsymbol{t}(\epsilon))$ belongs to $(V^{\otimes n})_{\boldsymbol{\lambda}}^{\mathrm{sing}}$. This vector is an eigenvector of the Bethe algebra $\mathcal{B}$, acting on $(\otimes_{s=1}^n V(t_s^{(0)}(\epsilon)))_{\boldsymbol{\lambda}}^{\mathrm{sing}}$, and we have $\mathcal{D}_{v(\epsilon)}^{\mathcal{B}} = \mathcal{D}_{X(\epsilon)}$, see Theorem 3.2.

The vector $v(\epsilon)$ algebraically depends on ϵ. Let $v(\epsilon) = v_0\,\epsilon^{a_0} + v_1\,\epsilon^{a_1} + \dots$ be its Puiseux expansion, where v_0 is the leading nonzero coefficient.

Theorem 4.1. *For a generic choice of the maps $F_1, \dots, F_k$, the vector $F(v_0)$ is nonzero. Moreover, $F(v_0)$ is an eigenvector of the Bethe algebra $\mathcal{B}$, acting on $(\otimes_{s=1}^k L_{\boldsymbol{\lambda}^{(s)}}(b_s))_{\boldsymbol{\lambda}}^{\mathrm{sing}}$, and $\mathcal{D}_{F(v_0)}^{\mathcal{B}} = \mathcal{D}^0$.*

Proof. For any element $B \in \mathcal{B}$, the action of B on the $U(\mathfrak{gl}_N[t])$-module $\otimes_{s=1}^{n} V(z_s)$ determines an element of $\mathrm{End}(V^{\otimes n})$, polynomially depending on $z_1, \dots, z_n$. Since for every nonzero ϵ, the vector $v(\epsilon)$ is an eigenvector of $\mathcal{B}$, acting on $(\otimes_{s=1}^{n} V(t_s^{(0)}(\epsilon)))_{\boldsymbol{\lambda}}^{\mathrm{sing}}$, and $\mathcal{D}_{v(\epsilon)}^{\mathcal{B}} = \mathcal{D}_{X(\epsilon)}$, the vector v_0 is an eigenvector of $\mathcal{B}$, acting on $(\otimes_{s=1}^{k} V(b_s)^{\otimes n_s})_{\boldsymbol{\lambda}}^{\mathrm{sing}}$, and $\mathcal{D}_{v_0}^{\mathcal{B}} = \mathcal{D}^0$.

The $\mathfrak{gl}_N[t]$-module $\otimes_{s=1}^{k} V(b_s)^{\otimes n_s}$ is a direct sum of irreducible $\mathfrak{gl}_N[t]$-modules of the form $\otimes_{s=1}^{k} L_{\boldsymbol{\mu}^{(s)}}$, where $|\boldsymbol{\mu}^{(s)}| = n_s,\ s = 1, \dots, k$. Since $\mathcal{D}^0 \in \Delta_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$, the vector v_0 belongs to the component of type $\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s)$. Therefore, for a generic choice of the maps $F_1, \dots, F_k$, the vector $F(v_0)$ is nonzero.

Since the map $F_1 \otimes \dots \otimes F_k$, see (4.1), is a homomorphism of $\mathfrak{gl}_N[t]$-modules, the vector $F(v_0)$ is an eigenvector of the Bethe algebra $\mathcal{B}$, acting on $(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s))_{\boldsymbol{\lambda}}^{\mathrm{sing}}$, and $\mathcal{D}_{F(v_0)}^{\mathcal{B}} = \mathcal{D}^0$. □

Remark. The direction of the vector v_0 can depend on the choice of the algebraic curve $X(\epsilon)$ in $\Omega_{\boldsymbol{\lambda}}$. However, Theorem 2.1 yields that the direction of the vector $F(v_0)$ does not depend on either the choice of the curve $X(\epsilon)$ or the choice of the maps $F_1, \dots, F_k$.

Given $\mathcal{D} \in \Delta_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$, denote by $w(\mathcal{D})$ the vector $F(v_0) \in (\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s))_{\boldsymbol{\lambda}}^{\mathrm{sing}}$ constructed from $\mathcal{D}$ in Section 4.2. The vector $w(\mathcal{D})$ is defined up to multiplication by a nonzero number. The assignment $\mathcal{D} \mapsto w(\mathcal{D})$ gives the correspondence, which is inverse to the correspondence $v \mapsto \mathcal{D}_v^{\mathcal{B}}$ in Theorem 2.1.

4.3. **Completeness of Bethe ansatz for $\mathfrak{gl}_N$ Gaudin model.** The construction of the vector $w(\mathcal{D}) \in (\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s))_{\boldsymbol{\lambda}}^{\mathrm{sing}}$ from a differential operator $\mathcal{D} \in \Delta_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ can be viewed as a (generalized) Bethe ansatz construction for the $\mathfrak{gl}_N$ Gaudin model, cf. the Bethe ansatz constructions in [Ba], [RV], [MV1], [MV2].

The following statement is contained in Theorem 6.1, Corollary 6.2 and Corollary 6.3 of [MTV4].

Theorem 4.2. *If $b_1, \dots, b_k$ are distinct real numbers, then the action of the Bethe algebra on $(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s))_{\boldsymbol{\lambda}}^{\mathrm{sing}}$ is diagonalizable and has simple spectrum.*

Hence, for generic complex numbers $b_1, \dots, b_k$, there exists an eigenbasis of the action of the Bethe algebra on $(\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s))_{\boldsymbol{\lambda}}^{\mathrm{sing}}$. This eigenbasis is unique up to permutation of vectors and multiplication of vectors by nonzero numbers.

Corollary 4.3. *If $b_1, \dots, b_k$ are distinct real numbers or $b_1, \dots, b_k$ are generic complex numbers, then the collection of vectors*

$$\{\, w(\mathcal{D}) \in (\otimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s))_{\boldsymbol{\lambda}}^{\mathrm{sing}} \mid \mathcal{D} \in \Delta_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}} \,\}$$

is an eigenbasis of the action of the Bethe algebra.

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